In routine practical analysis one must consider the relationship between economic requirements and the use of assumed functions with greater numbers of undetermined parameters than there are element node point displacements. Normally, the objective of engineering analysis is to obtain an acceptable result at minimum cost. The use of "excess" undetermined parameters may not only complicate the formulation but will certainly add expense to the computation for a given network size. Thus, it is possible that the simplest element relationships will provide acceptable results with the coarsest network one would choose to use, or else, with a larger network, at less cost. Note the closeness of the values of the terms in the stiffness matrix of Ref. 1 for 5 and 7 term stress assumptions, respectively; equations for the former, however, can be simply and explicitly formulated whereas equations for the latter are not readily formulated and otherwise require significant matrix operations.

Finally, a more complete exploitation of matrix structural analysis methods involves their use in instability, vibration, thermal stress, and inelastic analyses, among others. Consequently, it would be desirable for examinations of concepts in the formulation of discrete element relationships to include consideration of the appropriate terms for these phenomena.

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Reply by Author to R. H. Gallagher

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THE author wishes to point out that Eq. (23) of Ref. 1 has been chosen to insure the boundary compatibility between neighboring elements. For example, for two neighboring rectangular elements (I) and (II) as shown in Fig. 1, the displacements $u(y)$ along the edge BC of both elements are given by the same function

$$u_{BC}(y) = [1 - (y/b)]u_B + (y/b)u_C \quad (1)$$

where u_B and u_C are the horizontal displacements at corners B and C, respectively. It is seen that the edge displacement $u_{BC}(y)$ of either element is not affected, for example, by a vertical displacement at A or F.

If the displacement functions given by Eqs. (3) of the foregoing comment is employed, two different displacement functions will be resulted. They are, on element (I),

$$u_{BC}(y) = \left(1 - \frac{y}{b}\right) u_B - \frac{y}{b} u_C + \frac{y}{2a}\left(1 - \frac{y}{b}\right) \times (v_B - v_B + v_C - v) \quad (2)$$

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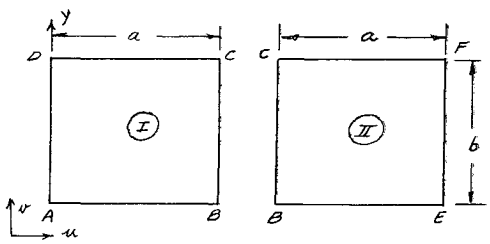

Fig. 1 Two neighboring rectangular elements.

and on element (II),

$$u_{BC}(y) = \left(1 - \frac{y}{b}\right) u_B + \frac{y}{b} u_C + \frac{y}{2a}\left(1 - \frac{y}{b}\right) \times (v_B - v_E + v_F - v_C) \quad (3)$$

Hence the two horizontal displacements will, in general, not be compatible because $(v_A - v_B - v_C - v_D)$ and $(v_B - v_E + v_F - v_C)$ are not correlated.

Thus, the author has a feeling that the numerical result given at the bottom of page 1335 of Ref. 1 are not identical to those which would be obtained by use of the equations formulated by Turner, et al., in Ref. 2.

The author is in full agreement with Gallagher that both accuracy and cost should be taken into consideration when a method is selected for a routine practical analysis, and hence there is a limit for the number of undetermined parameters to be used in formulating the stiffness matrix. The author, however, has been trying to find out whether there is basically a stiffness matrix that will yield results better than others when the same size networks are used.

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Comments on "Effect of Gas Composition on the Ablation Behavior of a Charring Material"

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A PAPER by Vojvodich and Pope[1] presented the results of an experimental investigation on the ablation behavior of a charring material. Chemical reactions were shown to occur between the material and an air environment, and the gas phase reactions were found to be more significant than predicted by the theory derived from the work of Cohen, Bromberg, and Lipkis and the work of Hartnett and Eckert.[2] A close examination of the experimental results reveals a possible discrepancy in the reported combustion effects. This observation follows from a thermochemical analysis of the ablative material and a discussion and analysis reported in Ref. 3. Additional clarification of the theory[2] and notation of the similarity to the theory of Lees[4] should enhance the interpretation of the experimental results.

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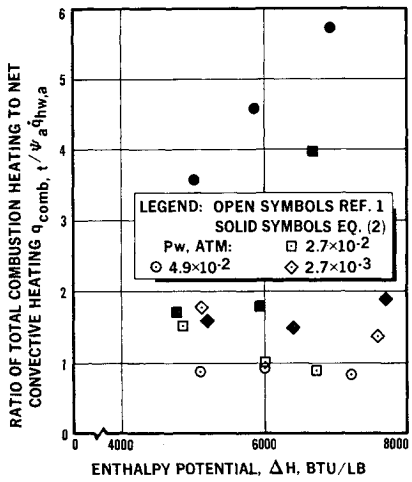

Fig. 1 Variation of combustion heating rate with enthalpy.

The experimental combustion heating results were determined from Eq. (A7) derived in the Appendix of Ref. 1†:

$$\frac{\dot{q}_{\text{comb},\,t}}{\psi_a \dot{q}_{hw,\,a}} = \frac{(\psi_n/\psi_a)\dot{q}_{hw,\,n} - \dot{q}_{hw,\,a}}{\dot{q}_{hw,\,a}} \tag{1}$$

However, the same results should be directly obtainable from Eq. (A2) after a slight rearrangement, i.e.,

$$\frac{\dot{q}_{\text{comb},\,t}}{\psi_a\, \dot{q}_{hw,\,a}} = \frac{\epsilon\sigma T_w^4 + [k_c(\partial T/\partial Y)]_w - \psi_a \dot{q}_{hw,\,a}}{\psi_a\, \dot{q}_{hw,\,a}} \tag{2}$$

where

$$[k_c(\partial T/\partial Y)]_w \cong \dot{m}_v h_A$$

The experimental information for estimating the magnitude of the combustion heating using Eq. (2) is presented in Ref. 1. Approximate results from Eq. (2) are shown in Fig. 1. These results were obtained using the following values given in Ref. 1: $\epsilon = 0.85$, $h_A = 1000$ Btu/lb, and $\dot{m}_v = 0.7$ times the reported vapor injection rates of Fig. 6c. The value of ψ that was used is the Swann and Pitman second-order approximation (Ref. 1, Fig. 11). In determining the mass transfer effects, the mass surface reaction rate was not included; for this reason, the results from Eq. (2) could be expected to be conservative. The results shown in Fig. 1

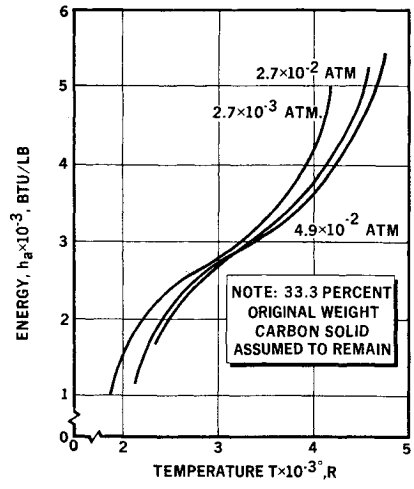

Fig. 2 Total energy of vapor products from phenolic nylon decomposition.

† The nomenclature is the same as that of Ref. 1.

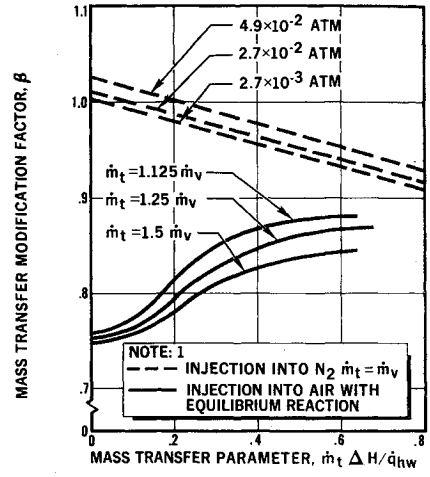

Fig. 3 Mass transfer modification factor for gas phase phenolic nylon decomposition.

are disquieting, since the values increase with boundary-layer enthalpy potential, and most of the data points are higher than those reported in Ref. 1.

It is not clear that the difference between the two equations can be explained by the experiments alone, but a thermochemical analysis of the ablation shows the additional considerations that must be included. A thermochemical equilibrium digital computer program predicts that the resin vapor products absorb between 1500 and 5500 Btu/lb (in decomposing and being heated by the char layer) for the experimental surface temperatures. These results are shown in Fig. 2. Since these vapor products are subject to combustion reactions, the mass-transfer factor would be expected to be different for the nitrogen and air tests. The analysis described in Ref. 3, which assumes chemical equilibrium at the char surface, is used to present the factor β as a function of the blowing parameter (Fig. 3). Char surface oxidation is included for the air environment.

Values from Figs. 2 and 3 and nitrogen test data from Ref. 1 were substituted into Eq. (A10) of Ref. 1. Approximate calculations indicate that the stagnation region vapor injection rates are less than 0.7 times the reported rates. Support for the calculations as opposed to the higher reported rates results from the fact that energy stored in the char layer will continue to decompose the resin after the end of a test and the fact that this energy is proportional to char thickness and surface temperature. Transient ablation characteristics should also be evaluated, since true steady-state ablation does not exist unless the surface recedes at the same rate as the resin decomposition zone.

Comparisons of the experimental results with theory can be extended and generalized. The theory of Hartnett and Eckert predicts that the increase in the heat flux to the surface due to gas-phase combustion above the value for injection of inert vapors is given by

$$\dot{q}_{\text{comb},\,v}/\psi_a\, \dot{q}_{hw,\,a} = K_{oe}\, h_c/\Delta H \tag{3}$$

where K_{oe} is the mass fraction of oxygen in the freestream, and h_c is the heat released per pound of oxygen consumed, provided all of the oxygen reacts. The low pressures, high stagnation enthalpies, and high surface temperatures indicate that the oxygen would be in the atomic state. The most plausible reactions that may occur are

$$C(S) + O \rightarrow CO$$

$$h_c = 10{,}000 \text{ Btu/lb}$$

and

$$C_2H_2 + 2O \rightarrow 2CO + H_2$$

$$h_c = 11{,}500 \text{ Btu/lb}$$

The limiting reaction is

$$2H + O \rightarrow H_2O$$

$$h_c = 25{,}000 \text{ Btu/lb}$$

The large number of reactions that may simultaneously occur for different ablation conditions limits the usefulness of Eq. (3) to qualitative comparisons.

The restrictions on the theory (that the reactions occur at a sharp flame front, that the location of the flame front is determined by the dimensionless release rate and that the release rate must be low) cannot be used to account for disagreement between theory and experiments of this type. The theory presented by Lees[4] accounts for boundary-layer combustion by assuming thermochemical equilibrium at the decomposing surface; Lees also indicates that the results are independent of endothermic or exothermic reactions within the boundary layer and external to the surface. This author developed an analysis for ablative material behavior[3] from Ref. 4 and independently carried through the algebraic manipulations that showed that Eq. (3) predicted essentially identical combustion effects. The Lees theoretical approach of Ref. 4 is general and is certainly applicable for stagnation region investigations. It is doubtful that the laminar boundary-layer equations would be invalid in low-pressure experimental stagnation region investigations of ablative materials where there is significant surface thermal reradiation. The theoretical analyses are, of course, dependent upon unity Prandtl, Schmidt, and Lewis numbers. However, experimental results are subject to inaccuracies and empirical curve fits of mass-transfer effects.

This author is of the opinion that the experimental results presented in Ref. 1 are misleading. The discrepancy that is shown between using Eqs. (1) and (2) deserves an explanation or at least a clarification since the reported experimental results of Ref. 1 would tend to invalidate the theory of Refs. 2–4. This author concludes that experiments of this type may be considered conclusive only if Eqs. (1) and (2) give essentially identical results. However, the refinement of the experimental technique should permit satisfactory evaluation of combustion effects predicted by theory.

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Reply by Authors to C. L. Arne

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THE preceding comments regarding Ref. 1 question the validity of the experiments and express concern over the lack of agreement between the results and existing theoretical predictions of combustion heating. Considering first the question of experimental techniques, it should be noted that the results obtained from the equation employed by the authors, i.e.,

$$\frac{\dot{q}_{\text{comb},\,t}}{\psi_a \dot{q}_{hw,\,a}} = \frac{\psi_n \dot{q}_{hw,\,n}}{\psi_a \dot{q}_{hw,\,a}} - 1 \qquad (1)\ddagger$$

should agree with the alternative approach suggested by Arne, namely,

$$\frac{\dot{q}_{\text{comb},\,t}}{\psi_a \dot{q}_{hw,\,a}} \approx \frac{\epsilon\sigma T_w^4 + \dot{m}_v h_A}{\psi_a \dot{q}_{hw,\,a}} - 1 \qquad (2)$$

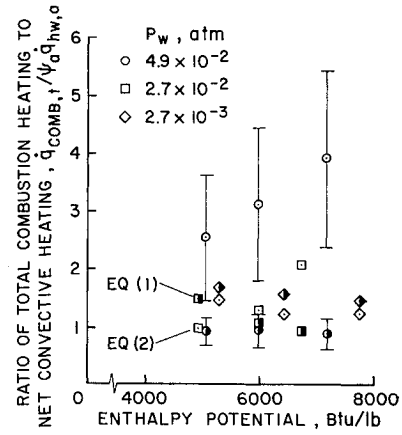

Fig. 1 Variation of total combustion heating rate with enthalpy: a comparison of Eq. (1) and (2).

Indeed, the comparison presented in Fig. 1 indicates satisfactory agreement between the two methods for all but the highest temperature conditions (corresponding to $P_w = 4.9 \times 10^{-2}$ atm) where the differences can be attributed in part to the large uncertainty levels inherent in Eq. (2). That is, the results obtained by the authors from Eq. (1) do not depend explicitly upon either the surface temperature T_w or the intrinsic heat capacity of the material h_A, but depend only upon the measurements of heating rates and a definition of the heat-transfer parameter ψ. On the other hand, the results computed from Eq. (2) are highly dependent upon surface temperature, particularly at high values of T_w where the reradiated energy from the ablator surface becomes the predominant mode of heat rejection. One then would expect that the reliability of Eq. (2) would be very sensitive to uncertainties in the temperature. The uncertainty in each variable appearing in Eqs. (1) and (2) was estimated, and the extent to which these individual uncertainties propagate into the results was calculated using the methods outlined in Ref. 2. These calculations are represented in Fig. 1 as the bands associated with the circular symbols. It is to be noted that the magnitudes of $\dot{q}_{\text{comb},\,t}/\psi \dot{q}_{hw,\,a}$ at $P_w = 4.9 \times 10^{-2}$ atm, as obtained by the authors from Eq. (2), are lower than the corresponding calculations of Arne which, as he has stated, are approximate. Although the total difference between the two techniques is not accounted for by uncertainty considerations alone, it is apparent that little, if any, significance may be attached to the differences cited by Arne at these conditions.

With regard to the discussion of the thermochemical analysis and role of the intrinsic heat capacity h_A in influencing the calculations, it is interesting to note that the adequate agreement of the two methods at low temperatures, mentioned previously, would no longer hold if the calculated values of h_A proposed by Arne in Fig. 2 were used in place of 1000 Btu/lb. Furthermore, recent measurements of h_A have been performed at Ames,[5] and in contrast to the calculations, indicate only a slight variation of h_A with temperature.

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‡ Nomenclature is that of Ref. 1.